On Acyclic Colorings of Graphs

Abu Reyan Ahmed, Md. Mazharul Islam, and Md. Saidur Rahman

Graph Drawing and Information Visualization Laboratory,
Department of Computer Science and Engineering
Bangladesh University of Engineering and Technology
abureyanahmed@yahoo.com, rmd.mazharulislam@yahoo.com, saidurrahman@cse.buet.ac.bd

Abstract. An acyclic coloring of a graph G is a coloring of the vertices of G, where no two adjacent vertices of G receive the same color and no cycle of G contains vertices of only two colors. An *acyclic k-coloring* of a graph G is an acyclic coloring of G using k colors. In this paper we show the necessary and sufficient condition of acyclic coloring of a complete k-partite graph. Then we derive the minimum number of colors for acyclic coloring of such graphs. We also show that any complete k-partite graph G having $n_1, n_2, ..., n_k$ vertices in its $P_1, P_2, ..., P_k$ partition respectively is acyclically $(2k-1)$-colorable using $\sum_{i \neq j, i,j \leq k} n_i n_j + n_{max} + (k-1) - \sum_{i=0}^{k-1} (k-i) n_{i+1}$ division vertices, where $n_{max} = max(n_1, n_2, ..., n_k)$. Finally we show that there is an infinite number of cubic planar graphs which are acyclically 3-colorable.

Keywords: Acyclic Coloring, Acyclic Chromatic Number, k-partite Graph, Cubic Planar Graph, Graph Subdivision.

1 Introduction

An *acyclic coloring* of a graph G is a coloring of G with no bichromatic cycle. An *acyclic k-coloring* of a graph G is an acyclic coloring of G using k colors. Acylic colorings of graphs find applications in diverse areas [9,10,11]. For example, an acyclic coloring of a planar graph has been used to obtain upper bounds on the volume of a 3-dimensional staright-line grid drawing of a planar graph [9]. Consequently, an acyclic coloring of a planar graph subdivision can give upper bounds on the volume of a 3-dimensional polyline grid drawing, where the number of division vertices gives an upper bound on the number of bends sufficient to achieve that volume. The acyclic chromatic number of a graph helps to obtain an upper bound on the size of "feedback vertex set" of a graph, which has wide applications in opareting system, database system, genome assembly, and VLSI chip design [10]. Acyclic colorings are also used in efficient computation of Hessian matrix [11].

The concept of acyclic coloring of a graph was introduced by Grunbaum [12] and is further studied in the last two decades in several works [2,1,7,5,6,3] among others. Grunbaum proved an upper bound of nine for the acyclic chromatic number of any planar graph G, with $n \geq 6$ vertices. Then Mitchem [16], Albertsorn and Berman [1], Kostochka [15] and finally Borodin [7] improved this upper bound to eight, seven, six and five respectively. Concerning the computational complexity of the corresponding decision problem, Kostochka [14] proved that deciding whether a planar graph admits an acyclic 3-coloring is *NP*-hard and Ochem [18] proved that the same holds for bipartite planar graphs of maximum degree 4. Bipartite graph is a k-partite graph with $k = 2$. Many practical problems such as networking, textile engineering [13] is directly related to k-partite graphs. While coloring complete k-partite graphs we have found an interesting property that although their chromatic number is always equal to k but their acyclic chromatic number is greater than or equal to k. We thus try to find a minimum bound for acyclic chromatic number of such graphs. In this paper we find out the minimum acyclic chromatic number of any complete k-partite graphs.

A k-subdivision of a graph G is a graph G' obtained by replacing every edge of G with a path that has at most k internal vertices. We call these internal vertices the *division vertices* of G. Wood [21] observed that every graph has a 2-subdivision that is acyclically 3-colorable. Angelini and Frati [4] proved that every triangulated planar graph with n vertices has a 1-subdivision with $3n - 6$ division vertices that is acyclically 3-colorable. This upper bound on the number of division vertices reduces to $2n - 6$ in the case of acyclic 4-coloring [17]. In this paper we show that any complete k-partite graph G having $n_1, n_2, ..., n_k$ vertices in its $P_1, P_2, ..., P_k$ partition respectively is acyclically $(2k - 1)$-colorable using $\sum_{i \neq j, i,j \leq k} n_i n_j + n_{max} + (k - 1) - \sum_{i=0}^{k-1} (k - i) n_{i+1}$ division vertices, where $n_{max} = max(n_1, n_2, ..., n_k)$.

Acyclic 3-coloring of a cubic planar graph is also an interesting open problem. In this paper we show that there are infinite number of cubic planar graphs which are acyclically 3-colorable.

In this paper we examine acyclic colorings of k-partite graphs and acyclic 3-

colorability of cubic planar graphs. Our results are as follows.

- In Section 3 we show the necessary and sufficient condition for acyclic coloring of any complete k-partite graphs. We also derive the minimum number of colors for acyclic coloring of such graphs.
- In Section 4 we reduce the number of colors using subdivision.
- In Section 5 we show that there are infinite number of planar cubic graphs which are acyclically 3-colorable. Note that although every cubic graph admits acyclic 4-coloring [19], every cubic graph does not always admit acyclic 3-coloring [4].

2 Preliminaries

In this section we present some definitions and preliminary results that are used throughout the paper.

A *graph* G is a tuple (V, E) which consists of a finite set V of vertices and a finite set E of edges; each edge is an unordered pair of vertices. We often denote the set of vertices of G by $V(G)$ and the set of edges by $E(G)$. A graph is *planar* if it can be embedded in the plane so that no two edges intersect geometrically except at a vertex to which they are both incident.

If the vertex set V of a graph G can be partitioned into k disjoint sets V_1,V_2,...,V_k in such a way that any edge of G joins a vertex of V_x to a vertex of V_y where $x \neq y$, then G is called *k-partite graph*. If every vertex of a partition is joined to every vertex of all other partion then G is called a *complete k-partite graph*.

Subdividing an edge (u, v) of a graph G is the operation of deleting the edge (u, v) and adding a path $u(= w_0), w_1, w_2, ..., w_k, v(= w_{k+1})$ through new vertices $w_1, w_2, ..., w_k, k \geq 1$, of degree two. A graph G' is said to be a *subdivision* of a graph G if G' is obtained from G by subdividing some of the edges of G. A vertex v of G' is called an *original vertex* if v is a vertex of G; otherwise, v is called a *division vertex*.

A *cubic graph* G is a graph such that every vertex of G has degree 3. If a cubic graph G is planar then G is a cubic planar graph.

3 Acyclic Coloring of Complete *k*-Partite Graphs

In this section we prove the necessary and sufficient condition of acyclic coloring of a complete k-partite graph. We also derive the minimum number of colors needed for acyclic coloring of such graphs.

Theorem 1. *Let G be a complete k-partite graph, then any proper coloring of G is an acyclic coloring if and only if there is at most one partition having two vertices of same color.*

Proof. *Necessity:* Assume for a contradiction that there are more than one partition of G having two vertices of same color. Let V_x and V_y be such two partitions. Let c_1 and c_2 be the repeated colors in partitions V_x and V_y, respectively. Then G contains a bichromatic cycle of colors c_1 and c_2, since G is a complete k-partite graph. This is a contradiction, since the coloring of G is acyclic.

Sufficiency: If there is no partition P having two vertices of same color then the proper coloring of G is also an acyclic coloring. We thus asume that there is a partition P having two vertices of same color. Then any cycle C that does not go through P has at least three vertices of different color. If a cycle goes through a vertex v in P, then the two neighbors u and w of v on C and v have different colors. Thus the coloring of G is acyclic. □

Theorem 1 immediately yields the following corollary.

Corollary 1. *Let G be a complete k-partite graph. Then the acyclic chromatic number of G is equal to $|V(G)| - x + 1$, where x is the size of the maximum partition.*

Proof. According to Theorem 1, vertices of at most one partition can have same color. Thus to color G acyclically with the minimum number of colors, all vertices of the maximum partition of G must be colored with same color. Otherwise the acyclic coloring would not be minimum. All other vertices of G must be colored with different colors. So the acyclic chromatic number of G is equal to $|V(G)| - x + 1$ where x is the size of the maximum partition. □

4 Acyclic Coloring with Subdivision

As we mentioned in Section 1 acyclic coloring of graph subdivisions has huge applications in theory and practice. In such applications it is desirable to use smaller number of division vertices. In Section 3 we provided some properties about acyclic coloring of complete k-partite graphs. In this section we deal with acyclic coloring of subdivisions of complete k-partite graphs. One can easily get an acyclic coloring of a complete k-partite graph using $(2k-1)$ colors, by adding one division vertex on each edge of the graph. This naive approch will add

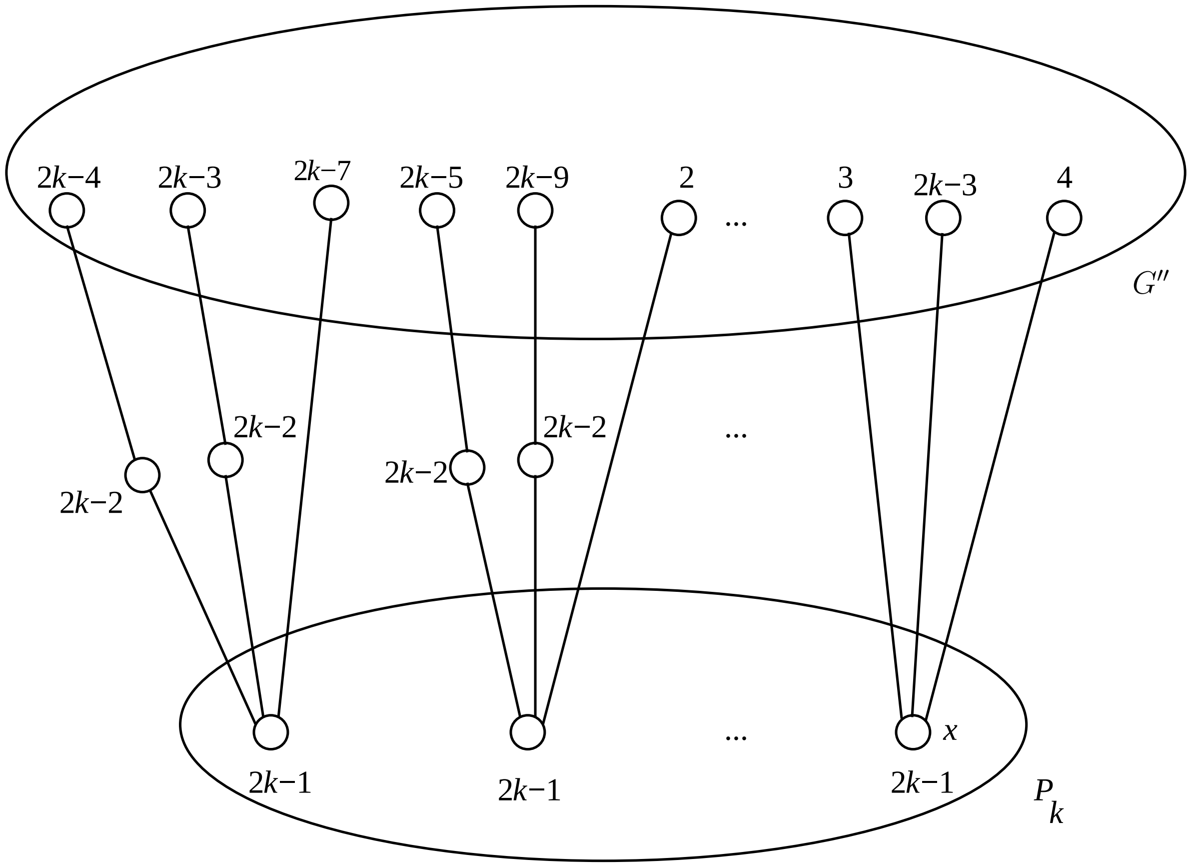

Fig. 1. Structure of graph G'.

$\sum_{i \neq j, i,j \leq k} n_i n_j$ division vertices. We can reduce this number of division vertices by a careful observation as in the following theorem.

Theorem 2. *Let G be a complete k-partite graph having $n_1, n_2, ..., n_k$ vertices in its $P_1, P_2, ..., P_k$ partition, respectively. Then G has a subdivision G' which is acyclically $(2k-1)$-colorable using $\sum_{i \neq j, i,j \leq k} n_i n_j$ $+ n_{max} + (k-1) - \sum_{i=0}^{k-1} (k-i) n_{i+1}$ division vertices, where $n_{max} = max(n_1, n_2, ..., n_k)$.*

Proof. We denote by $G_l, 1 \leq l \leq k$, the subgraph of G induced by $P_1 \cup P_2 \cup .... \cup P_l$. Then $G_k = G$. We can assume $n_1 \geq n_2 \geq ... \geq n_k$; otherwise we can reorder the partition in this way.

We prove the claim by induction on l. When $l = 1$, we can use $2l - 1 = 2.1 - 1 = 1$

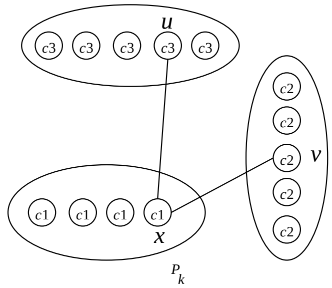

Fig. 2. Vertices u and v belongs to two different partitions.

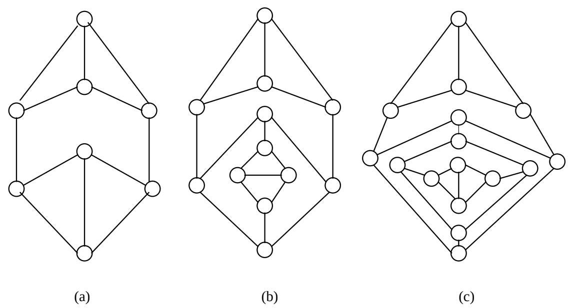

Fig. 3. Recursive construction of infinite graphs which are not acyclically 3-colorable.

color and $\sum_{i\neq j, i,j\leq l} n_i n_j \; + n_{max} + (l-1) - \sum_{i=0}^{k-1}(l-i)n_{i+1} = \sum_{i\neq j, i,j\leq 1} n_i n_j \; + n_1 + (1-1) - \sum_{i=0}^{1-1}(1-i)n_{i+1} = 0 + n_1 + 0 - n_1 = 0$ division vertices to get an acyclic coloring of P_1. Since there exists no edge between two vertices of same partition, we can color all vertices of P_1 with the first color. Hence the above condition satisfies, P_1 is acyclically 1-colorable using no division vertices.

We thus assume that $l > 1$ and that the claim is true for graphs $G_1, G_2, G_3, ..., G_l$ where $1 \leq l \leq k-1$. We now have to show that the claim is also true for G_k.

We first obtain G_{k-1} by deleting P_k from G_k. By induction hypothesis, G_{k-1} has an subdivision G'_{k-1} which is acyclically $2(k-1)-1 = (2k-3)$ colorable, where the number of division vertices is equal to $\sum_{i\neq j, i,j\leq k-1} n_i n_j$ $+n_{max} + (k-2) - \sum_{i=0}^{k-2}(k-1-i)n_{i+1}$. We now obtain a graph G^* by adding the deleted edges from all vertices in P_k to all original vertices in G'_{k-1}. Let x be an arbitary vertex of P_k in G^*. Now for each vertex $y \in P_k - x$, we subdivide $d(y) - 1$ edges

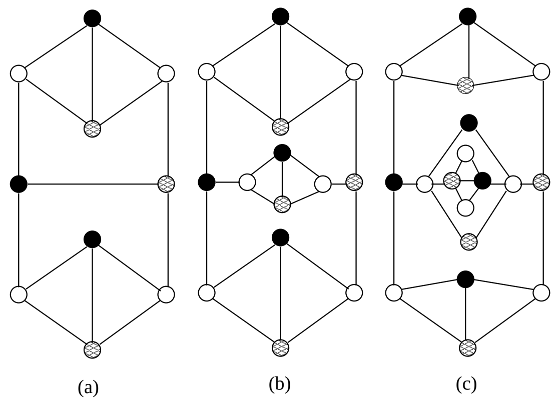

Fig. 4. Recursive construction of acyclically 3-colorable graphs.

incident to y by replacing each edge with path containing one division vertex. Note that we do not subdivide exactly one edge incident to y. We now color the newly division vertices with $(2k-2)^{th}$ color and we color all vertices of P_k with $(2k-1)^{th}$ color. Let G' be the resulting graph as illustrated in Fig. 1. Clearly, G' is a subdivision of G_k which is colored with $(2k-1)$ colors. Now to complete the proof it remains to show that G' is acyclically colored using $\sum_{i \neq j, i,j \leq k} n_i n_j$ $+ n_{max} + (k-1) - \sum_{i=0}^{k-1} (k-i) n_{i+1}$ division vertices.

We first prove that G' is acyclically colored. Any cycle that does not go through P_k contains vertices of at least three colors according to induction hypothesis. If a cycle goes through $P_k - x$ then it also contains vertices of at least three colors since it must contain one division vertex, as illustrated in Fig. 1. Let us consider a cycle C that goes through x. If two neighbours u and v of x on C belong to two different partitions then C contains vertices of at least three colors, as illustrated in Fig. 2. If u and v belong to same partition P, then C must have a vertex w where $w \notin (P \cup x)$. If $w \in (P_k - x)$ then C contains vertices of at least three colors, as we mentioned above. If $w \notin (P \cup P_k)$ then C must traverse at least three partitions. Hence C must have at least three colors. Thus C is always acyclic. Hence G' is acyclically colored too.

The number of edges incident to P_k is equal to $n_k(n_1 + n_2 + ... + n_{k-1})$. Number of edges incident to $P_k - x$ and not subdivided is equal to $(n_k - 1)$. Number of edges incident to x is equal to $(n_1 + n_2 + ... + n_{k-1})$. So number of subdivision vertices which was not present in G'_{k-1}, but present in G' is equal to $n_k(n_1+n_2+...+n_{k-1})-(n_k-1)-(n_1+n_2+...+n_{k-1})$. So total number of subdivision vertices in G' is equal to number of subdivision vertices in $G'_{k-1}+n_k(n_1+n_2+...+n_{k-1})-(n_k-1)-(n_1+n_2+...+n_{k-1}) = \sum_{i\neq j,i,j\leq k} n_in_j + n_{max}+(k-1)-\sum_{i=0}^{k-1}(k-i)n_{i+1}$. So G has a subdivision G' which is acyclically $(2k-1)$-colorable using $\sum_{i\neq j,i,j\leq k} n_in_j$ $+n_{max}+(k-1)-\sum_{i=0}^{k-1}(k-i)n_{i+1}$ division vertices. □

5 Acyclic Coloring of Cubic Planar Graphs

A graph G is cubic if every vertex of G has degree exactly three. Cubic graphs arises in different kind of real world problems. Cubic graphs have been deeply investigated in the literature due to their appliacation in topology, 1-dimensional CW complex and polyhedra. According to Brooks's theorem [8] every cubic graph other than the complete graph K_4 can be colored with at most three colors. According to Vizing's theorem [20] every cubic graph needs either three or four colors for an edge coloring.

Acyclic coloring of cubic graphs have many interesting properties. Grunbaum proved that any cubic graph admits an acyclic 4-coloring. So it became an interesting problem whehter they also admits acyclic 3-coloring. Recently Frati [4] showed that there exists an infinite number of cubic planar graphs which admit no acyclic 3-coloring as showed in Fig. 3. In this section we will show an interesting opposite result. We found that there are infinite number of cubic planar graphs which admits acyclic 3-coloring.

Theorem 3. *There is an infinite number of cubic planar graphs which are acyclically 3-colorable.*

Proof. The graph in Fig. 4(a) is a cubic planar graph and acyclically 3-colorable. Now we take any edge and substitute it with a subgraph according to Fig. 4(b). There is no bichromatic cycle in the subgraph and any cycle passing through that subgraph must visit three colors. The new graph will also be planar, cubic and acyclically 3-colorable. In this way we can add this subgraph in any edges and produce an infinite number of cubic planar acyclically 3-colorable graphs. □

6 Conclusion

In this paper we have shown several results on the acyclic colorability of complete k-

partite graphs. We find the minimum chromatic number of complete k-partite graphs. Using subdivision this cromatic number can be reduced. Actually this reduction process is a two way optimization porcess. One way is to reduce the subdivision vertices and the other way is to reduce the number of colors. Here we reduce this cromatic number using subdivision. If G is a complete k-partite graph having $n_1, n_2, ..., n_k$ vertices in its $P_1, P_2, ..., P_k$ partition, respectively. Then G has a subdivision G' which is acyclically $(2k-1)$-colorable using $\sum_{i \neq j, i,j \leq k} n_i n_j + n_{max} + (k-1) - \sum_{i=0}^{k-1} (k-i) n_{i+1}$ division vertices, where $n_{max} = max(n_1, n_2, ..., n_k)$. Finally we show that there is an infinite number of cubic plananr graphs which are acyclically 3-colorable.

References

1. Albertson, M., Berman, D.: Every planar graph has an acyclic 7-coloring. Israel Journal of Mathematics 28, 169–174 (1977), 10.1007/BF02759792
2. Alon, N., Mcdiarmid, C., Reed, B.: Acyclic coloring of graphs. Random Structures & Algorithms 2(3), 277–288 (1991)
3. Alon, N., Sudakov, B., Zaks, A.: Acyclic edge colorings of graphs. J. Graph Theory 37(3), 157–167 (Jul 2001)
4. Angelini, P., Frati, F.: Acyclically 3-colorable planar graphs. J. Comb. Optim. 24(2), 116–130 (Aug 2012)
5. Borodin, O.V., Fon-Der Flaass, D.G., Kostochka, A.V., Raspaud, A., Sopena, E.: Acyclic list 7-coloring of planar graphs. J. Graph Theory 40(2), 83–90 (Jun 2002)
6. Borodin, O.V., Kostochka, A.V., Raspaud, A., Sopena, E.: Acyclic coloring of 1-planar graphs. diskretnyi analizi issledivanie operacii, series 1 6(4):2035 (1999)
7. Borodin, O.V.: On acyclic colorings of planar graphs. Discrete Mathematics 306(10-11), 953–972 (2006)
8. Brooks, R.L.: On colouring the nodes of a network. Cambridge Journals 37, 194–197 (1941)
9. Dujmovic, V., Morin, P., Wood, D.: Layout of graphs with bounded tree-width. SIAM journal of computing 34, 553–579 (2005)
10. Fertin, G., Gordard, E., Raspaud, A.: Miniumum feedback vertex set and acyclic color. Inf. Process. Lett 84(3), 131–139 (2002)
11. Gebremedhin, A., Tarafdar, A., Pothen, A., Walther, A.: Efficient computation of sparse hessians using coloring and automatic differentiation. INFORMS J Comput 21(2), 209–223
12. Grunbaum, B.: Acyclic Coloring of planar Graphs. Isreal Journal of Mathmatics 14, 390–408 (1973)
13. Grunert, T., Irnich, S., Zimmermann, H.J.: Cliques in k-partite graphs and their application in textile engineering
14. Kostochka, A.: Upper bounds of chromatic functions of graphs. PhD thesis
15. Kostochka, A.: Acyclic 6-coloring of planar graphs. Diskretn. Anal. 28, 40–56 (1976)

16. Mitchem, J.: Every planar graph has an acyclic 8-coloring. Duke Mathematical Journal 41(1), 177–181 (1974)
17. Mondal, D., Nishat, R.I., Whitesides, S., Rahman, M.S.: Acyclic colorings of graph subdivisions revisited. Journal of Discrete Algorithms 16(0), 90 – 103 (2012)
18. Ochem, P.: Negative results on acyclic improper colorings. Felsner S (ed) European conference on combinatorics, graph theory and applications (EuroComb 05). DMTCS AE, 357–362 (2005)
19. Skulrattanakulchai, S.: Acyclic colorings of subcubic graphs. Information Processing Letters 92(4), 161–167 (2004)
20. Vizing, V.G.: On an estimate of the chromatic class of a p-graph. Diskret. Analiz. 3, 25–30 (1964)
21. Wood, D.: Acyclic, star and oriented colorings of graph subdivisions. Discrete Mathematics & Theoretical Computer Science 7(1), 37–50 (2005)